Section 13.7 Exercise 3

(a) $Q = a \ln(L+1)$

P price of commodity

L units of labor

w wage rate

$$\pi(L) = PQ - wL$$
$$= P(a \ln(L+1)) - wL$$
$$= aP \ln(L+1) - wL$$
$$\pi'(L) = aP \frac{1}{L+1}(1) - w$$
$$= aP(L+1)^{-1} - w$$

Solve for L
$$aP\left(\frac{1}{L+1}\right) - w = 0$$
$$\frac{aP}{L+1} = w$$
$$aP = wL + w$$
$$\frac{aP - w}{w} = L$$
$$\boxed{\frac{aP}{w} - 1 = L^*}$$

↳ maximizes profits

L^* depends on $\underbrace{a, P, w}_{\text{parameters.}}$

$$\pi''(L) = aP(-1)(L+1)^{-2}$$
$$= -\frac{aP}{(L+1)^2} < 0$$

(b) Verify the envelope theorem!

$$\pi(L^*, \underbrace{P, w, a}_{\text{parameters}}) = \pi^*(P, w, a)$$

↑ original profit function ↑ optimal profits

explicitly recognize that profits depend on a, P, w.

Envelope Theorem
$$\boxed{\frac{\partial \pi^*}{\partial P}} = \frac{\partial \pi(L^*, P, w, a)}{\partial P} = \left.\frac{\partial \pi(L, P, w, a)}{\partial P}\right|_{L=L^*}$$

equal?

$$\pi(L) = aP \ln(L+1) - wL$$
$$L^* = \frac{aP}{w} - 1$$
$$\pi^* = \boxed{aP} \ln(\boxed{L^*}+1) - w\boxed{L^*} = aP \ln\left(\frac{aP}{w} - 1 + 1\right) - w\left(\frac{aP}{w} - 1\right)$$
$$= aP \ln\left(\frac{aP}{w}\right) - aP + w$$
$$= aP \ln aP - aP \ln w + aP + w$$
$$= \boxed{aP \ln a + aP \ln P - aP \ln w - aP + w}$$

deriv wrt P

$$\frac{\partial \pi^*}{\partial P} = aP\left(\frac{1}{L^*+1}\right)\left(\frac{\partial L^*}{\partial P}\right) + a\ln(L^*+1) - w\frac{\partial L^*}{\partial P}$$

$$= \underbrace{\left[ap\left(\frac{1}{L^*+1}\right) - w\right]}_{= 0 \text{ (first-order conditions)}} \frac{\partial L^*}{\partial p} + a\ln(L^*+1)$$

$$= \underbrace{a\ln(L^*+1)}_{= Q^* \text{ (optimal quantity produced)}}$$

$$\pi(L) = ap\ln(L+1) - wL$$

no L^* here, because we are looking at original profit function

$$\frac{\partial \pi}{\partial p} = a\ln(L+1)$$

$$\left.\frac{\partial \pi}{\partial p}\right|_{L=L^*} = a\ln(L^*+1) = Q^*$$

Sec 13.7 Example 3 & Exercise 4

$$\frac{\partial \pi^*}{\partial p} = Q^*$$

$$\frac{\partial \pi^*}{\partial w} = -L^*$$

$$\frac{\partial \pi^*}{\partial r} = -K^*$$

$$\frac{\partial Q^*}{\partial r} \overset{?}{=} -\frac{\partial K^*}{\partial p}$$

$$\frac{\partial Q^*}{\partial w} \overset{?}{=} -\frac{\partial L^*}{\partial p}$$

$$\frac{\partial L^*}{\partial r} \overset{?}{=} \frac{\partial K^*}{\partial w}$$

Young's Theorem: $z = f(x,y) \Rightarrow z''_{xy} = z''_{yx}$

$$\boxed{\frac{\partial \pi^*}{\partial p} = Q^*} \Rightarrow \frac{\partial^2 \pi^*}{\partial r \partial p} = \boxed{\frac{\partial Q^*}{\partial r}} \Rightarrow \frac{\partial Q^*}{\partial r} = -\frac{\partial K^*}{\partial p}$$

Take deriv. wrt r.

$$= \frac{\partial}{\partial p}\left(\frac{\partial \pi^*}{\partial r}\right)$$

$= -K^*$ from Example 3

Ch8 functions of one variable $\max/\min f(x)$, $x \in D$

Ch13 " " two variables or more $\max/\min f(x,y)$, $(x,y) \in S$

} unconstrained opt.

Ch14 " " two or more variables $\max/\min f(x,y)$ subject to $\boxed{g(x,y) = c}$ ✓

Constrained opt.

equality constraint

$\max/\min f(x,y)$
subject to $\boxed{g(x,y) \leq c}$

inequality constraint

Section 14.1 Example 1 Substitution method works

2 could also work but have to choose carefully

Section 14.3 Example 1 no way that you would use subst method

Section 14.1 Example 2 λ is not something out of nowhere

$\Rightarrow$ may have some meaning / interpretation

Section 14.1 Example 3 $\rightarrow$ general version of Example 1)

$U(x,y) = xy$
subject to $2x + y = 100$

$U(x,y) = Ax^a y^b$
subject to $px + qy = m$

Take $A = 1$, $a = 1$, $b = 1$
$p = 2$, $q = 1$, $m = 100$.

$$x^* = \frac{a}{a+b}\frac{m}{p} \qquad 0 < \frac{a}{a+b} < 1$$

$$y^* = \frac{b}{a+b}\frac{m}{q} \qquad 0 \leq \frac{b}{a+b} < 1$$

$$\boxed{px^*} = \boxed{\frac{a}{a+b} m}$$

How much is the bill for buying x^*

$$\underbrace{qy^*}_{} = \boxed{\frac{b}{a+b} m}$$

how much is the bill for buying y^*

Section 14.1 Ex 9
Section 14.2 Ex 5

$$U(x,y) = \alpha \ln(x-a) + \beta \ln(y-b)$$

$$\text{alg.} = \ln(x-a)^{\alpha} + \ln(y-b)^{\beta}$$

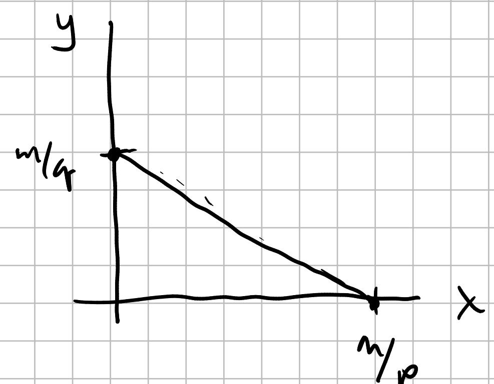
$$u(x,y) = \ln (x-a)^{\alpha}(y-b)^{\beta}$$

$$\exp U(x,y) = (x-a)^{\alpha}(y-b)^{\beta}$$

Section 14.1 Example 1

$$\frac{MU_x}{MU_y} = \frac{P_x}{P_y} \quad \text{or} \quad \frac{MU_x}{P_x} = \frac{MU_y}{P_y}$$

$px + qy = m$ (line)



$$U(x,y)$$

$$U(x,y) = \underline{1}$$