8.6 → multiple stationary points

8.7 → issue encountered in Section 8.6

understand behavior of functions which we could not necessarily visualize

Ch 8 $f(x)$
one variable

Ch 13, 14 $f(x,y) \sim f(x,y,z)$ harder to visualize

$\pi(Q) = R(Q) - C(Q)$

$R(Q)$: revenue as a function of Q

$C(Q)$: costs as a function of Q

aiming for a "low-resolution" visualization (curve sketching)

local = relative

global = absolute

Compare definition of local max/min with max/min in Section 8.1

Section 8.6 Example 1

$$f(x) = \frac{1}{9}x^3 - \frac{1}{6}x^2 - \frac{2}{3}x + 1$$

$$f'(x) = \frac{1}{3}x^2 - \frac{1}{3}x - \frac{2}{3} = \frac{1}{3}(x^2 - x - 2) = \boxed{\frac{1}{3}(x-2)(x+1)}$$

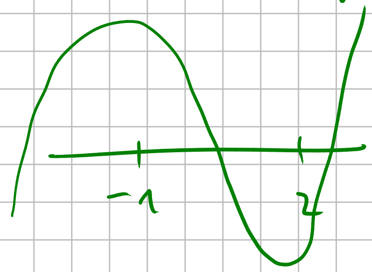
$$f'(x) = 0 \Rightarrow \frac{1}{3}(x-2)(x+1) = 0$$

$x = 2$ or $x = -1$ (two stationary points)

-1 $\quad$ 2 $\quad$ $\mathbb{R}$

Test points		Sign of $f'(x)$	Remarks
-2	$x < -1$	$(+)(-)(-) = (+)$	f is increasing
	$x = -1$		f has a local max
0	$-1 < x < 2$	$(+)(-)(+) = (-)$	f is decreasing
	$x = 2$		f has local min
3	$x > 2$	$(+)(+)(+) = (+)$	f is increasing

↓ not the exact graph



$$f(x) = -\frac{1}{9}x^3 + \frac{1}{6}x^2 + \frac{2}{3}x - 1$$

$$f'(x) = -\frac{1}{3}x^2 + \frac{1}{3}x + \frac{2}{3} = -\frac{1}{3}(x-2)(x+1) = 0$$

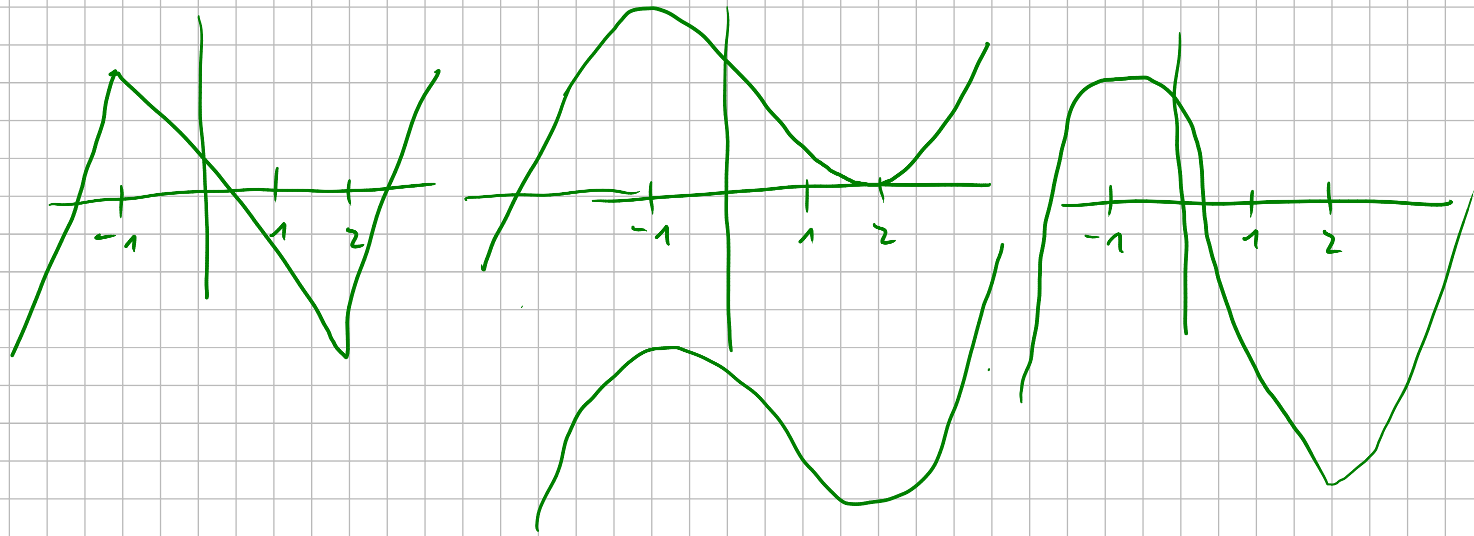
Bad habit / portion! not $f'(x)$

$(x-2)(x+1) = 0$

$\Rightarrow x = 2, x = -1$

$f'(x) = (x-2)(x+1)$

Separate what the first derivative looks like & the calculation of stationary pts



Thm 8.6.2

(c) If $f'(c)=0$ & $f''(c)=0$, then ?

case where second derivative test fails.

SDT does not have a conclusion. Try a different test.

connected to 8.7

$f(x) = \frac{1}{9}x^3 - \frac{1}{6}x^2 - \frac{2}{3}x + 1$ Sketch the graph

$f'(x) = \frac{1}{3}(x-2)(x+1)$ stationary points $x=2$, $x=-1$

$f''(x) = \frac{2}{3}x - \frac{1}{3} = \frac{1}{3}(2x-1)$ $\ast$ a point where 2nd derivative is equal to zero

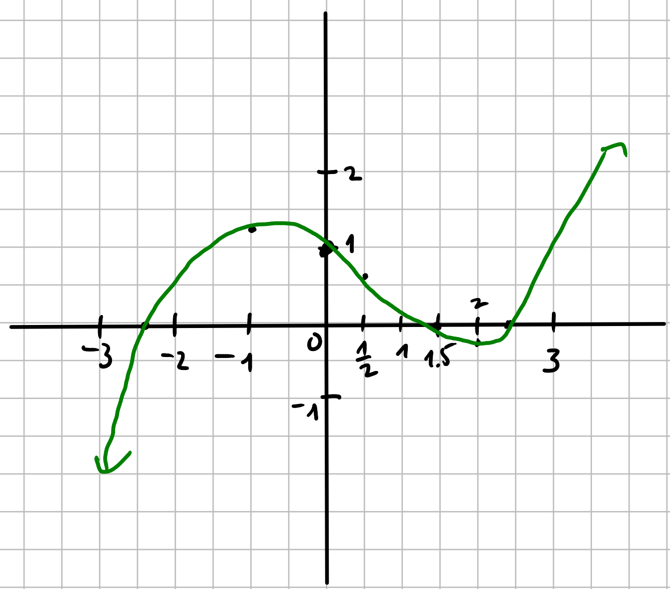
$\frac{1}{3}(2x-1) = 0 \Rightarrow x = \frac{1}{2}$

$-1 \qquad \frac{1}{2} \qquad 2 \qquad \mathbb{R}$

$f'(x) = \boxed{\frac{1}{3}}(x-2)(x+1)$

$f''(x) = \frac{1}{3}(2x-1)$

Test pts		sign of f'	sign of f''	Remarks.
-2	$x<-1$	$(+)(-)(-)=(+)$	$(+)(-)=(-)$	f is increasing, f is concave
	$x=-1$			f has a local max
0	$-1<x<\frac{1}{2}$	$(+)(-)(+)=(-)$	$(+)(-)=(-)$	f is decreasing – f is concave
	$x=\frac{1}{2}$			no extreme pt, f has an inflection point Section 8.7
1	$\frac{1}{2}<x<2$	$(+)(-)(+)=(-)$	$(+)(+)=(+)$	f is decreasing – f is convex
	$x=2$			f has a local min
3	$x>2$	$(+)(+)(+)=(+)$	$(+)(+)=(+)$	f is increasing – f is convex



$f(x) = \frac{1}{9}x^3 - \frac{1}{6}x^2 - \frac{2}{3}x + 1$

$f(x) = 0$ when
- $x = 1.5$
- $x = -2.44949$
- $x = 2.44949$

when will f intersect x-axis?

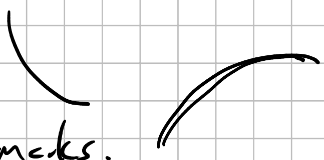
$f(0) = 1$

when will f intersect y-axis

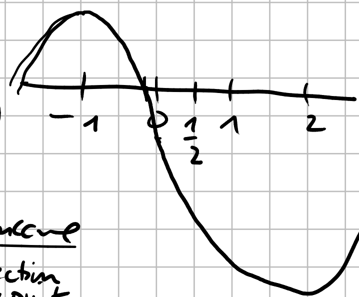
$f(-1) = 1.388\ldots$

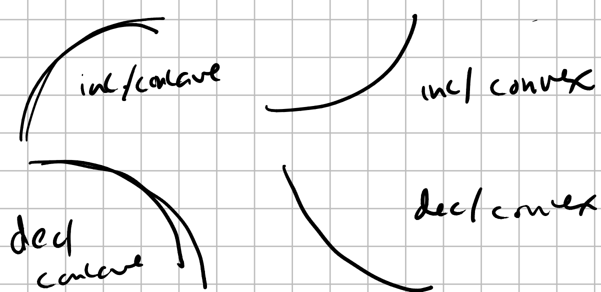
$f(\frac{1}{2}) = 0.638F\ldots$

$f(2) = -0.1111\ldots$



	Sign of f'	Sign f''	Remarks.
$x < -1$	$(+)(-)(-)=(+)$	$(+)(-)=(-)$	f is increasing, f is concave
$x = -1$			f has a local max
$-1 < x < \frac{1}{2}$	$(+)(-)(+)=(-)$	$(+)(-)=(-)$	f is decreasing – f is concave
$x = \frac{1}{2}$			no extreme pt, f has an inflection point
$\frac{1}{2} < x < 2$	$(+)(-)(+)=(-)$	$(+)(+)=(+)$	f is decreasing – f is convex
$x = 2$			f has a local min
$x > 2$	$(+)(+)(+) =(+)$	$(+)(+)=(+)$	f is increasing – f is convex





Thm 8.7.1

(a) c is an inflection pt of f $\Rightarrow$ $f''(c) = 0$

$\not\Leftarrow$

Reminds of: c is a (local) max pt or (local) min pt of f $\Rightarrow$ $f'(c) = 0$

Review Problems

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

① $f(x) = \dfrac{x^2}{x^2+2}$

$$f'(x) = \frac{(x^2+2)(2x) - x^2(2x)}{(x^2+2)^2}$$

$$= \frac{2x[x^2+2-x^2]}{(x^2+2)^2} = \frac{2x(2)}{(x^2+2)^2} = \frac{4x}{(x^2+2)^2}$$

$$f''(x) = \frac{(x^2+2)^2(4) - \boxed{4}x(2)(\boxed{x^2+2})(2x)}{(x^2+2)^4}$$

$$= \frac{4(x^2+2)[x^2+2 - 2x(2x)]}{(x^2+2)^4} = \frac{4(x^2+2)(-3x^2+2)}{(x^2+2)^4}$$

$$= \frac{4(-3x^2+2)}{(x^2+2)^3}$$

Stationary points: $\dfrac{4x}{(x^2+2)^2} = 0 \Rightarrow x = 0$

candidate Inflection points: $\dfrac{4(-3x^2+2)}{(x^2+2)^3} = 0 \Rightarrow -3x^2+2=0$

$\Rightarrow x^2 = \frac{2}{3}$

$\Rightarrow x = \pm\sqrt{\frac{2}{3}}$

$-\sqrt{\frac{2}{3}}$ $\quad 0 \quad$ $\sqrt{\frac{2}{3}}$ $\quad \mathbb{R}$

Test points		Sign of f'	Sign of f''	Remarks
-1	$x < -\sqrt{\frac{2}{3}}$	$\frac{(+)(-)}{(+)} = (-)$	$\frac{(+)(-)}{(+)} = (-)$	f is dec., f is concave
-0.81	$x = -\sqrt{\frac{2}{3}}$			not an extreme pt. f has an inflection pt
-0.5	$-\sqrt{\frac{2}{3}} < x < 0$	$\frac{(+)(-)}{(+)} = (-)$	$\frac{(+)(+)}{(+)} = (+)$	f is dec., f is convex
	$x = 0$			f has a local min
0.5	$0 < x < \sqrt{\frac{2}{3}}$	$\frac{(+)(+)}{(+)} = (+)$	$\frac{(+)(+)}{(+)} = (+)$	f is inc., f is convex
	$x = \sqrt{\frac{2}{3}}$			not an extreme pt., f has an inflection pt.
1	$x > \sqrt{\frac{2}{3}}$	$\frac{(+)(+)}{(+)} = (+)$	$\frac{(+)(-)}{(+)} = (-)$	f is inc., f is concave

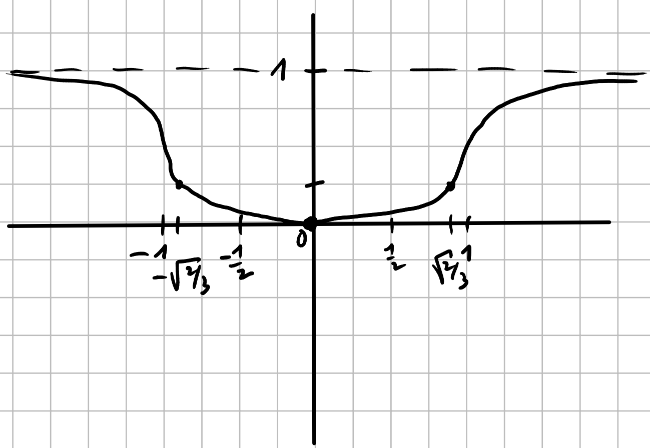
f is decreasing when $x < 0$

f is increasing when $x > 0$

$$\lim_{x \to \infty} \frac{x^2}{x^2+2} = ? \qquad \lim_{x \to \infty} \frac{1}{1+\frac{2}{x^2}} = 1$$

$$\lim_{x \to -\infty} \frac{x^2}{x^2+2} = ? \qquad \lim_{x \to -\infty} \frac{1}{1+\frac{2}{x^2}} = 1$$

$$\frac{x^2}{x^2+2} = \frac{\frac{x^2}{x^2}}{\frac{x^2+2}{x^2}} = \frac{1}{1+\frac{2}{x^2}} \quad \text{(algebra)}$$



$$f(0) = \frac{0}{0+2} = 0$$

$$f\left(-\sqrt{\tfrac{2}{3}}\right) = \frac{2/3}{2/3+2} = \frac{\frac{2}{3}}{\frac{8}{3}} = 1/4$$

$$f\left(\sqrt{\tfrac{2}{3}}\right) = \frac{2/3}{2/3+2} = \frac{1}{4}$$