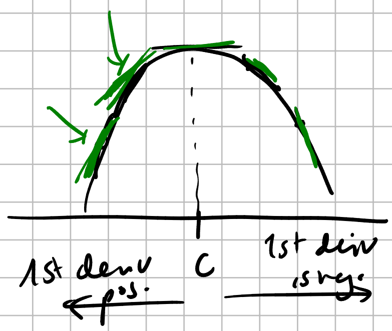
First derivative test → looking at sign variation of $f'(x)$

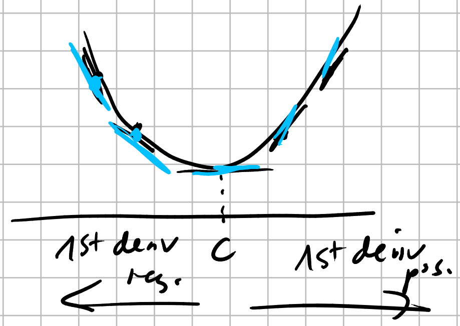
Thm 6.2.2 → looking at the magnitude of $f'(x)$ and how it relates to the 2nd derivative $f''(x)$.

↳ first derivative of the first derivative

f is increasing $\Leftrightarrow f'(x) \geq 0$

f' is increasing $\Leftrightarrow f''(x) \geq 0$





1st derivative is decreasing in magnitude

$f''(x) \leq 0$

f is Concave

1st derivative is increasing in magnitude.

$f''(x) \geq 0$

f is convex

Two options for checking max/min
- Sign variation of 1st derivative
- Calculate 2nd deriv. (check the sign)

↳ are only for cases where you only have one stationary point in an interval

Section 6.2 Exercise 5

$g(x) = x^3 \ln x$ Find possible extreme pts

$x \in (0, \infty)$

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

$$g'(x) = x^3\left(\frac{1}{x}\right) + (\ln x)(3x^2)$$

$$= x^2 + 3x^2 \ln x$$

$$= x^2(1 + 3\ln x)$$

stationary points: $x^2(1+3\ln x) = 0$

$\cancel{x = 0}$ (ruled out)

$\ln 0$ does not make sense (undefined)

$$1 + 3\ln x = 0$$

$$3\ln x = -1$$

$$\ln x = -1/3$$

$$\exp(\ln x) = \exp\left(-\frac{1}{3}\right)$$

$$x = e^{-\frac{1}{3}}$$

Options: use FDT or show that g is either concave or convex

Concave/convex: $g''(x) = x^2\left(\frac{3}{x}\right) + (1+3\ln x)(2x)$

$= 3x + 2x(1+3\ln x)$

$= x[3+2+6\ln x]$

$x \in (0,+\infty)$ $\quad = \underbrace{x}_{>0}\,[\underbrace{5+6\ln x}]$

$5+6\ln x$ is not always negative (positive) over $x\in(0,\infty)$

We could not say for sure that $g(x)$ is concave or convex over $(0,\infty)$.

Try out points $x=1 \Rightarrow 5+6\ln 1 = 5$

$x = 0.1 \Rightarrow 5+6\ln(0.1) \approx -8.8$

You know that $x = e^{-1/3}$ is stationary point. It is possible to adjust I in Thm 1.2.2 but it would not answer the question posed in the exercise!

Better option is to use FDT!

$x>1 \Rightarrow \ln x > 0$
$0<x<1 \Rightarrow \ln x < 0$

$0 < x < e^{-\frac{1}{3}} \approx 0.7165$ $\quad g'(x) = \underset{(+)}{\boxed{x^2}}\underset{(-)}{(1+3\ln x)} < 0$ $\quad 1+3(-1)<0$, $1+3(-0.25)>0$

Test pt $= 0.5$

$x > e^{-1/3}$ $\quad g'(x) = \underbrace{x^2}_{(+)}\underbrace{(1+3\ln x)}_{(+)} > 0$

Test pt: 1

$\Rightarrow x = e^{-1/3}$ is a minimum pt, by the first derivative test.

If you know (comfortable) with algebra,

$(1+3\ln x) > 0$

$\Leftrightarrow \quad 3\ln x > -1$

$\Leftrightarrow$ increasing $\quad \ln x > -\frac{1}{3}$

$\Leftrightarrow \quad \exp(\ln x) > \exp(-\frac{1}{3})$

$\Leftrightarrow \quad x > e^{-1/3}$

Whenever $x > e^{-1/3} \rightarrow 1+3\ln x > 0$

Whenever $1+3\ln x > 0 \rightarrow x > e^{-1/3}$

$\frac{d}{dx}[e^{x}] = e^{x}$

$f(x) = I_0 + kx + Ae^{-\alpha x}$

$f'(x) = k + A\boxed{e^{-\alpha x}}(-\alpha) = k - A\alpha e^{-\alpha x}$

$k + Ae^{-\alpha x}(-\alpha) = 0$ $\quad$ Solve for x

$$-A\alpha e^{-\alpha x} = -k$$

$$e^{-\alpha x} = \frac{k}{A\alpha}$$

$$\ln e^{-\alpha x} = \ln \frac{k}{A\alpha}$$

$$-\alpha x = \ln \frac{k}{A\alpha}$$

$$x = -\frac{1}{\alpha} \ln \frac{k}{A\alpha} = -\frac{1}{\alpha} \ln\left(\frac{A\alpha}{k}\right)^{-1}$$

$$= \frac{1}{\alpha} \ln \underbrace{\left(\frac{A\alpha}{k}\right)}_{>0}$$

because $\alpha > 0$, $A\alpha > k$

Options to check min:
FOT, concave/convex

$$f''(x) = -A\alpha e^{-\alpha x}(-\alpha)$$

$$= \underbrace{A}_{>0}\underbrace{\alpha^2}_{>0} \underbrace{e^{-\alpha x}}_{>0} > 0$$

$\Rightarrow$ f is concave over $(0, \infty)$

By Thm 6.2.2, $x_0 = +\frac{1}{\alpha}\ln\left(\frac{A\alpha}{k}\right)$ is a minimum pt of f.
($\frac{1}{\alpha}\ln\left(\frac{A\alpha}{k}\right) > 0$)

(b) no optimization but you will be doing comparative statics

What happens to an optimum / an equilibrium if a parameter which influences the optimum changes?

$$x_0 = \frac{1}{\alpha}\ln\left(\frac{A\alpha}{k}\right) \Rightarrow \frac{\partial x_0}{\partial \alpha} = ? \text{ sign? how large?}$$

$$\frac{\partial x_0}{\partial k} = ? \text{ sign? how large?}$$

$$\frac{\partial x_0}{\partial A} = ? \text{ sign? how large?}$$

Here the optimum is available in closed form

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profits = revenues − costs
(revenues = price × quantity)

π not 3.1416...
↳ Greek letter π/pi

$$\max_N \pi(N) = p\,Y(N) - qN$$

$Y(N)$: some function of N / function is unspecified

If the firm were profit-maximizing at $N = N^*$, then we should see that $\pi'(N^*) = 0$

$$pY'(N^*) - q = 0 \Rightarrow pY'(N^*) = q$$

$$\pi'(N^*) = pY'(N^*) - q$$

$Y(N) \to Y$ is a function of N

$$\pi'(N) = pY'(N) - q$$
$$\pi''(N) = \underbrace{p}_{>0}\, \underbrace{Y''(N)}_{} \leq 0$$

production function should have diminishing marginal products

$P(Q)$ P as a function of Q (general)

$$\pi(Q) = Q P(Q) - kQ$$
$$\pi'(Q) = QP'(Q) + P(Q)\cdot 1 - k$$

If the monopolist were profit maximizing at $Q = Q^*$, then

$$\pi'(Q^*) = 0 \Rightarrow Q^* P'(Q^*) + P(Q^*) - k = 0$$
$$\Rightarrow \boxed{Q^* P'(Q^*) + P(Q^*) = k}$$

You do not exactly know what $Q^* = ?$

$$\frac{\partial Q^*}{\partial k} = ? \quad \text{But}$$

Compare with dikes problem.

You need an indirect way of obtaining this derivative even if you don't know what Q^* looks like!

Q^* should satisfy $\to$ $Q^* P'(Q^*) + P(Q^*) = k$
we also know that Q^* will depend on what k is.

Take the derivative of the first-order condition with respect to k.

chain rule

$$\frac{d}{dk}\left[\overset{Q^*(k)}{Q^*} \overset{Q^*(k)}{P'(Q^*)} + P(Q^*)\right] = \frac{d}{dk}[k]$$

$$\boxed{Q^*} P''(Q^*) \frac{dQ^*}{dk} + P'(Q^*)\frac{dQ^*}{dk} + P'(Q^*)\frac{dQ^*}{dk} = 1$$

implicit differentiation

$$\boxed{\frac{dQ^*}{dk}}\left(Q^* p''(Q^*) + p'(Q^*) + p'(Q^*)\right) = 1$$

$$\frac{dQ^*}{dk} = \frac{1}{Q^* p''(Q^*) + 2p'(Q^*)}$$

$$\frac{d\Pi(Q^*)}{dk} = \frac{d}{dk}\left[Q^* P(Q^*) - kQ^*\right] = \ldots$$

(c) connected to Section 13.7