Def of inverse

The inverse of $A_{(n\times n)}$ is the matrix X such that $AX = XA = I$.

unique $= A^{-1}$

Thm 1.6.1

(a) A is invertible (given)

A^{-1} is also invertible, $(A^{-1})^{-1} = A$ (to show)

To show that A^{-1} is invertible, we have to find a matrix X such that $A^{-1}X = I$ or $XA^{-1} = I$. How to find X?

Since it is given that A is invertible, by definition,

$A^{-1}A = AA^{-1} = I$ is true. We actually found that X!

$A^{-1}X = I$

$A^{-1}A = I$

$AA^{-1} = I$

Therefore, $(A^{-1})^{-1} = A$.

(b) A & B are invertible (given)

AB is invertible, $(AB)^{-1} = B^{-1}A^{-1}$ (to show)

To show that AB is invertible, I have to find a matrix X such that $(AB)\boxed{X} = I$ or $X(AB) = I$.

Since A is invertible, $AA^{-1} = I$ or $A^{-1}A = I$

Since B is invertible, $BB^{-1} = I$ or $B^{-1}B = I$

$\boxed{A}\boxed{B}\boxed{X} = I$ Multiply both sides by A^{-1} (on the left)

$\underbrace{A^{-1}A}BX = \underbrace{A^{-1}I}$

$\underbrace{IB}X = A^{-1}$

$BX = A^{-1}$

$\underbrace{B^{-1}B}X = B^{-1}A^{-1}$ Multiply both sides by B^{-1} (on the left)

$\underbrace{IX} = B^{-1}A^{-1}$

$X = B^{-1}A^{-1}$

We found an $X = B^{-1}A^{-1}$ such that $(AB)(B^{-1}A^{-1}) = I$.

This means that $(AB)^{-1} = B^{-1}A^{-1}$

or $(B^{-1}A^{-1})^{-1} = AB$

Section 16.6 Exercise 9 $\Leftarrow$ $X'X$ is invertible

$\underset{m\times n}{X}$ $|X'X| \neq 0$ (given)

$A^2 = A$

Show that $I_m - X(X'X)^{-1}X'$ is idempotent.

To show that $I_m - X(X'X)^{-1}X'$ is idempotent,

$$(I_m - X(X'X)^{-1}X')(I_m - X(X'X)^{-1}X') \stackrel{?}{=} I - X(X'X)^{-1}X'$$

$$\underbrace{X}_{m\times n}(\underbrace{\underbrace{X'}_{n\times m}\underbrace{X}_{m\times n}}_{n\times n})^{-1}\underbrace{X'}_{n\times m}$$

$$\overset{m\times m}{I_m} - X(X'X)^{-1}X')(\overset{m\times m}{I_m} - \overset{m\times n}{X}(X'X)^{-1}X')$$

$$= I_m - X(X'X)^{-1}X' - X(X'X)^{-1}X'$$
$$+ (X(X'X)^{-1}X')(X(X'X)^{-1}X')$$

$$= I_m - 2X(X'X)^{-1}X' + X(X'X)^{-1}X'X(X'X)^{-1}X'$$

$$X I_n (X'X)^{-1}X'$$

$$X(X'X)^{-1}X'$$

scalar

$A = 1 \cdot A$

$$= I_m - 2X(X'X)^{-1}X' + 1X(X'X)^{-1}X'$$

$$= I_m - 1 \cdot X(X'X)^{-1}X' \Rightarrow \text{idempotent}$$

Section 16.6 #10

$A = (\quad)$

$B = (\quad)$

$C = (\quad)$

$D = (\quad)$

$$\overset{(2\times 2)}{\underset{(2\times 3)}{A}\underset{(3\times 2)}{B}} + \overset{(2\times 2)}{C\underset{(2\times 2)}{X}} = \underset{(2\times 2)}{D}$$

Must realize that X has to be a 2×2 matrix.

Find X?

one option $AB+CX=D$

$$\begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix} & \\ & \end{pmatrix}+\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}x_1&x_2\\x_3&x_4\end{pmatrix}=\begin{pmatrix} & \\ & \end{pmatrix}$$

other option $AB+CX=D$

$$\boxed{C}X=D-AB$$

C is invertible

Be careful here. We need to make sure that C^{-1} exists.

C is invertible because $|C|=4-6=-2\neq 0$.

C^{-1} exists!

$$CX=D-AB$$

$$\Rightarrow \underbrace{C^{-1}C}_{I}X=C^{-1}(D-AB)$$

$$X=C^{-1}(D-AB)$$

You complete the exercise

Section 16.6 Exercise 11

$$\underset{\substack{\| \\ C\cdot C}}{C^2}+C=I$$

Be careful here.

$$C\cdot C+C=C(?+?)$$

$$=C(C+I)$$

Observe that $C^2+C=I \Rightarrow C(C+I)=I$

$\Rightarrow$ $C+I$ is the inverse of C. (by def)

another way to write this is $C^{-1}=\underbrace{C+I}_{I+C}$

Section 16.6 #5(a) last question.

$A^3-2A^2+A-I_3=0$ (can be shown by direct calculation)

Use $\nearrow$ to find A^{-1}.

$$A^3-2A^2+A-I_3=0 \Rightarrow A^3-2A^2+A=I_3$$

$$\boxed{A}\underbrace{(A^2-2A+I_3)}=I_3$$

Therefore, $A^{-1} = A^2 - 2A + I$
$= (A - I)(A - I)$

$x + y = 0$
$x - y = 0$

$x = 0,\ y = 0$ trivial

$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $\quad |A| = -2 \neq 0$

The only solution is the trivial solution.

$x + y = 0$
$2x + 2y = 0$

$x = 0, y = 0$ is a solution

$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ $\quad |A| = 0$

other non-trivial solutions exist

$(t, -t),\ t \in \mathbb{R}$

$(I_n - A)x = b$ $\quad (I_n - A)^{-1}$ exists

$(I_n - A)^{-1}(I_n - A)\, x = (I_n - A)^{-1} b$
$x = (I_n - A)^{-1} b$

Review Problems Ch 16
#8

#8 (a) you do this

$U = \begin{pmatrix} 1 & 1 & 1 & \cdots \\ 1 & \ddots & 1 & 1 \\ 1 & \cdots & & 1 \end{pmatrix}$ all ones

$(I_n + aU)(I_n + bU) = I_n + (a + b + nab)\,U$

Use $\uparrow$ to find inverse of $A = \begin{pmatrix} 4 & 3 & 3 \\ 3 & 4 & 3 \\ 3 & 3 & 4 \end{pmatrix}$

$= \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= 3\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\underbrace{\quad}_{3U}\ \underbrace{+I}$

Observe that $A = I_3 + 3U$. Find A^{-1}

$$(I_n + aU)(I_n + bU) = I_n + (a+b+nab)U$$

$$\underbrace{(I_3 + 3U)}_{A}\underbrace{\left(I_3 + \left(-\tfrac{3}{10}\right)U\right)}_{A^{-1}} = \underbrace{I_3 + (3+b+3\cdot 3\cdot b)U}_{I}$$

$3+b+3\cdot3\cdot b \to 0$

$AA^{-1} = I$

The only way for $I_3 + (3+b+3b)U$ to be I_3 only is when

$3+b+9b = 0$

$10b = -3$

$b = -\frac{3}{10}$

$3 - \frac{3}{10} + 3\cdot 3\left(-\frac{3}{10}\right)$

If $n=3$, $a=3$, $b=-\frac{3}{10}$

$a+b+nab$

$$(I_n + aU)(I_n + bU) = I_3 + 0\cdot U$$

Therefore, $A^{-1} = I_3 - \frac{3}{10}U = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \frac{3}{10}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$

$=$

$$A = \begin{pmatrix}4&3&3\\3&4&3\\3&3&4\end{pmatrix}$$

$U\cdot U =$

$U = \begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$

$U\cdot U = \begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$

$= \begin{pmatrix}3&3&3\\3&3&3\\3&3&3\end{pmatrix}$

$= 3U$

$$nU = U + \underbrace{U + \dots +}_{n \text{ times}} U$$

$$U^n = U\cdot U\cdot U \cdots U$$

$UU = \begin{pmatrix}1&\cdots&1\\ \vdots&\ddots&\vdots\\1&\cdots&1\end{pmatrix}\begin{pmatrix}1&\cdots&1\\ \vdots&\ddots&\vdots\\1&\cdots&1\end{pmatrix}$

$= \begin{pmatrix}n&\cdots&n\\ \vdots&\ddots&\vdots\\n&\cdots&n\end{pmatrix}$

$= n\begin{pmatrix}1&\cdots&1\\ \vdots&\ddots&\vdots\\1&\cdots&1\end{pmatrix} = nU$

$(I_n + aU)(I_n + bU)$

$= I_n + bU + aU + ab\underline{U^2}$

$= I_n + bU + aU + abnU$

$= I_n + (b+a+nab)U$