AB is not always defined

$AB \neq BA$

main differences between matrix multiplication & ordinary multiplication

$2 \cdot 5 = 5 \cdot 2$

Matrix multiplication is NOT commutative.

Section 15.3 Example 3 → dynamical systems

What would be the new market shares after a year?

$$\begin{array}{lccc}
 & \text{left from A} & \text{taken from B} & \text{taken from C} \\
\text{new market for A} & 0.85(0.2) + & 0.1(0.6) + & 0.1(0.2) = \underline{0.25} \\
B & 0.05(0.2) + & 0.55(0.6) + & 0.05(0.2) = \underline{0.35} \\
C & 0.10(0.2) + & 0.35(0.6) + & 0.85(0.2) = \underline{0.40}
\end{array}$$

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3
\end{aligned}$$

$$\underbrace{\begin{pmatrix} 0.85 & 0.10 & 0.10 \\ 0.05 & 0.55 & 0.05 \\ 0.10 & 0.35 & 0.85 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.6 \\ 0.2 \end{pmatrix}}_{\text{multiplying 2 matrices}} = \begin{pmatrix} 0.25 \\ \\ \\ \end{pmatrix}$$

$$\underbrace{\overset{A}{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}}}_{\text{coef matrix}} \underbrace{\overset{x}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}}_{\text{unknowns}} = \underbrace{\overset{b}{\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}}}_{\text{constants}}$$

market dynamics → Will things stabilize in the long run?

Ts mkt share after 1 yr.

$T(Ts)$ mkt share after 2 yrs. $T(Ts) = TTs = T^2 s$

T matrix

⋮

$\lim_{n\to\infty} T^n s$ = finite?

T transition matrix (model dynamics)

economics T (movements from employed to unemployed)

Systems of linear equations → matrix form

$$\underset{(m\times n)}{A}\,\underset{(n\times 1)}{x} = \underset{(m\times 1)}{b}$$

* Matrix multiplication is "roughly" a composition of linear transformations (linear functions)
* Matrices are "roughly" linear functions.

$$z_1 = a_{11} y_1 + a_{12} y_2 + a_{13} y_3$$
$$z_2 = a_{21} y_1 + a_{22} y_2 + a_{23} y_3$$

$$\underset{2\times 1}{\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}} = \underset{2\times 3}{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}} \underset{3\times 1}{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}}$$

Let T be a function where

$$T\left(\underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}}_{\text{input}}\right) = \underbrace{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}}_{\text{output}}$$

$$y_1 = b_{11} x_1 + b_{12} x_2$$
$$y_2 = b_{21} x_1 + b_{22} x_2$$
$$y_3 = b_{31} x_1 + b_{32} x_2$$

$$\underset{3\times 1}{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}} = \underset{3\times 2}{\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}} \underset{2\times 1}{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}$$

$$S\left(\underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\text{Input}}\right) = \underbrace{\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\text{Output}}$$

$$S\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \underbrace{\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}}_{\mathbf{B}}\underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\mathbf{x}}$$

$$T\left(\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) = \underbrace{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}}_{\mathbf{A}}\underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}}_{\mathbf{y}}$$

output

$S(\mathbf{x}) = \mathbf{B}\mathbf{x}$ (3×1)

$T(\mathbf{y}) = \mathbf{A}\mathbf{y}$ (2×1)

$$\underbrace{T(S(\mathbf{x}))}_{\text{composition}} = \underbrace{\boxed{\mathbf{AB}}}_{\text{matrix multiplication!}}\mathbf{x}$$

$$\begin{tikzcd} \overset{\text{input}}{\mathbf{x}} \arrow[r, "\text{Apply } S"'] \arrow[rr, bend right, "T(S(\mathbf{x}))"'] & \overset{\text{input}}{\mathbf{B}\mathbf{x}} \arrow[r, "\text{Apply } T"'] & \mathbf{AB}\mathbf{x} \end{tikzcd}$$

Section 16.9 Example 1

α alpha

β beta

γ gamma

Section 12.11 Example 1

$$5u + 5v = 2x - 3y$$
$$2u + 4v = 3x - 2y$$

Systems of 2 lin equats in 4 unknowns.

we are not necessarily interested in the solution itself but we are interested in how the solution is affected by a change in x (for example).

$$y = f(x)$$

$$\underbrace{dy}_{\text{approx chge in } y} = \underbrace{f'(x)}_{\text{per unit chge in } x}\overbrace{dx}^{\text{approx chge in } x}$$

$$5\,du + 5\,dv = 2\,dx - 3\,dy$$
$$2\,du + 4\,dv = 3\,dx - 2\,dy$$

Assume that only x changes $(dx \neq 0)$ but y does not $(dy = 0)$. How will u and v be affected?

$$5\,du + 5\,dv = 2\,dx$$
$$2\,du + 4\,dv = 3\,dx$$

$$\left.\begin{aligned} 5\frac{du}{dx} + 5\frac{dv}{dx} &= 2 \\ 2\frac{du}{dx} + 4\frac{dv}{dx} &= 3 \end{aligned}\right\} \quad \underset{A}{\begin{pmatrix} 5 & 5 \\ 2 & 4 \end{pmatrix}} \overset{*}{\begin{pmatrix} \frac{du}{dx} \\ \frac{dv}{dx} \end{pmatrix}} = \underset{b}{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}$$

* Calculus + linear algebra!

A lot of time was spent setting up / recognizing situations for which we have systems of linear equations!

How to solve? Section 15.6

Systematic way → Gaussian elimination

Example 1

$$\begin{aligned} 2x_2 - x_3 &= -7 \\ x_1 + 2x_2 + 3x_2 &= 2 \\ -3x_1 + 2x_2 + 2x_3 &= -10 \end{aligned} \quad \Rightarrow A = \begin{pmatrix} 0 & 2 & -1 \\ 1 & 2 & 3 \\ -3 & 2 & 2 \end{pmatrix} \quad b = \begin{pmatrix} -7 \\ 2 \\ -10 \end{pmatrix}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

augmented matrix

$$\begin{array}{c} A \mid b \\ \left(\begin{array}{ccc|c} 0 & 2 & -1 & -7 \\ 1 & 2 & 3 & 2 \\ -3 & 2 & 2 & -10 \end{array}\right) \end{array}$$

goal is to hopefully reach the following pattern

$$\left(\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right)$$

*'s could be anything.

NOTE but this pattern is not always going to happen.

$$\left(\begin{array}{ccc|c} 0 & 2 & -1 & -7 \\ 1 & 1 & 3 & 2 \\ -3 & 2 & 2 & -10 \end{array}\right) \xrightarrow[R_1 \leftrightarrow R_2]{} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 2 & -1 & -7 \\ -3 & 2 & 2 & -10 \end{array}\right)$$

exchange row 1 with row 2

new row 2 is half of the old row 2

$$R_2 \leftarrow \frac{1}{2} R_2 \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & -1/2 & -7/2 \\ -3 & 2 & 2 & -10 \end{array}\right)$$

$$\begin{array}{cccc} \cancel{3} & 3 & 9 & 6 \\ \cancel{-3} & 2 & 2 & -10 \\ \hline & 5 & 11 & -4 \end{array}$$

$R_3 \leftarrow 3R_1 + R_3$ or $R_1 + \frac{1}{3} R_3$ or $R_3 + 3R_1$

new row 3 is obtained by multiplying row 1 by 3 and adding to row 3

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & -1/2 & -7/2 \\ 0 & 5 & 11 & -4 \end{array}\right)$$

$R_3 \leftarrow -5R_2 + R_3$ or $\frac{1}{5} R_3 - R_2$

$$\begin{array}{cccc} 0 & -5 & \frac{5}{2} & \frac{35}{2} \\ 0 & 5 & 11 & -4 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & -1/2 & -7/2 \\ 0 & 0 & 27/2 & 27/2 \end{array}\right)$$

Group HW tomorrow at Neocities

grouping will be available at animospace (tomorrow)