$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & -1 \\ 1 & 2 & 5 \end{pmatrix}$$ Find $|A|$.

How to apply Rule F correctly?

The value of the determinant is unchanged if a multiple of one row is added to a different row.

I will start from

$$\begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & -1 \\ 1 & 2 & 5 \end{vmatrix} \underset{\substack{R_2 \leftrightarrow R_3 \\ \text{Rule D}}}{=} - \begin{vmatrix} 2 & 1 & 3 \\ 1 & 2 & 5 \\ 1 & 0 & -1 \end{vmatrix}$$ this is correct.

$$\underset{R_2 \leftarrow -\frac{1}{2}R_1 + R_2}{=} - \begin{vmatrix} 2 & 1 & 3 \\ 0 & \frac{3}{2} & \frac{7}{2} \\ 1 & 0 & -1 \end{vmatrix}$$ this is now correct.

$$\begin{matrix} -1 & -\frac{1}{2} & -\frac{3}{2} \\ 1 & 2 & 5 \end{matrix}$$

$$\underset{R_3 \leftarrow -\frac{1}{2}R_1 + R_3}{=} - \begin{vmatrix} 2 & 1 & 3 \\ 0 & \frac{3}{2} & \frac{7}{2} \\ 0 & -\frac{1}{2} & -\frac{5}{2} \end{vmatrix}$$ This is now correct.

$$\begin{matrix} -1 & -\frac{1}{2} & -\frac{3}{2} \\ 1 & 0 & -1 \end{matrix}$$

$$\underset{R_3 \leftarrow \frac{1}{3}R_2 + R_3}{=} - \begin{vmatrix} 2 & 1 & 3 \\ 0 & \frac{3}{2} & \frac{7}{2} \\ 0 & 0 & -\frac{4}{3} \end{vmatrix} = -(2)\left(\frac{3}{2}\right)\left(-\frac{4}{3}\right) = 4$$

$$\begin{matrix} 0 & \frac{1}{2} & \frac{7}{6} \\ 0 & -\frac{1}{2} & -\frac{5}{2} \end{matrix}$$

Why is my application of Rule F incorrect earlier in class?

Consider what I have done: Replace R_2 with $-2R_2 + R_1$ on a 2×2 matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Apply $R_2 \leftarrow -2R_2 + R_1$ to get

$$\begin{vmatrix} a_{11} & a_{12} \\ -2a_{21}+a_{11} & -2a_{22}+a_{12} \end{vmatrix} = a_{11}(-2a_{22}+a_{12}) - a_{12}(-2a_{21}+a_{11})$$

$$= -2a_{22}a_{11} + \cancel{a_{11}a_{12}} + 2a_{12}a_{21} - \cancel{a_{12}a_{11}}$$

$$= -2(a_{11}a_{22} - a_{12}a_{21})$$

Clearly, the result is <u>not</u> the same as $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$!

$= a_{11}a_{22} - a_{12}a_{21}$

Therefore, I applied Rule F incorrectly!

A correct application would be $R_2 \leftarrow R_2 - 2R_1$ for example.

Is $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ equal to $\begin{vmatrix} a_{11} & a_{12} \\ -2a_{11}+a_{21} & -2a_{12}+a_{22} \end{vmatrix}$?

$$= a_{11}(-2a_{12}+a_{22}) - a_{12}(-2a_{11}+a_{21})$$

$$= -\cancel{2a_{11}a_{12}} + a_{11}a_{22} + \cancel{2a_{12}a_{11}} - a_{12}a_{21}$$

$$= a_{11}a_{22} - a_{12}a_{21}$$

<u>Yes!</u>

This is also a big difference between the handling of row operations in Section 15.6 and Theorem 16.4.1 Rule F.